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1982 J. Phys. A: Math. Gen. 15 3367

(http://iopscience.iop.org/0305-4470/15/10/038)

J. Phys. A: Math. Gen. 15 (1982) 3367–3372. Printed in Great Britain

COMMENT

A two-parameter Bäcklund transformation for the Boussinesq equation

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Received 13 April 1982, in final form 25 May 1982

Abstract. A two-parameter Bäcklund transformation for the Boussinesq equation, $u_{tt}-u_{xx}-3(u^2)_{xx}-u_{xxxx}=0$, was obtained by using Hirota's bilinear operator. One can use the two arbitrary parameters in this Bäcklund transformation to derive the soliton solution, nonlinear superposition formula and infinitely many conservation laws.

Bäcklund transformation (BT) plays an important role in the study of the nonlinear evolution equation. It is intimately connected with the applicability of the inverse scattering method for solving the initial value problem, the existence of soliton solutions and infinitely many conservation laws, as well as the capability for the equation to transform to the full integrable Hamilton system. In this paper, we derive a two-parameter BT for the Boussinesq equation using Hirota's bilinear operator and show it is the two arbitrary parameters that play the key role when one hopes to obtain the soliton solution, nonlinear superposition formula and the infinite number of conservation laws of the equation using this BT.

The Boussinesq equation

$$u_{tt}-u_{xx}-3(u^2)_{xx}-u_{xxxx}=0 \tag{1}$$

is an important wave equation, first studied in 1872 by Boussinesq, in describing shallow-water wave propagation in both directions. Recently, it was found very useful in one-dimensional lattices, in the ion–acoustic waves in plasma and in the electric power transmission lines with nonlinear capacitance.

Equation (1) can be written as

$$(D_t^2-D_x^2-D_x^4)f\cdot f=0 \tag{2}$$

under the transformation

$$u=2(\log f)_{xx} \tag{3}$$

where Hirota's bilinear operators D_x and D_t are defined by

$$D_x^nD_t^m a(x,t)\cdot b(x,t)\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m a(x,t)b(x',t')|_{x=x',t=t'}. \tag{4}$$

0305-4470/82/103367+06\$02.00

Using Hirota's method, a bilinear BT can easily be derived (Hirota 1980)

$$(D_t + aD_x^2)f' \cdot f = 0 \tag{5a}$$

$$(aD_tD_x + D_x + D_x^3)f' \cdot f = 0 \tag{5b}$$

where $a^2 = -3$, f' and f are two solutions of equation (2). This BT does not have any arbitrary parameters. To derive the infinitely many conservation laws, the nonlinear superposition formula of solutions and the soliton solutions, one often has to use the arbitrary parameters contained in the BT. We now derive a two-parameter BT for equation (2).

First we introduce some properties of the bilinear operator which we shall use in the following.

Lemma 1.

$$D_z^m a \cdot b = (-1)^m D_z^m b \cdot a \tag{6a}$$

$$D_z^m a \cdot a = 0 \qquad \text{for odd } m \tag{6b}$$

where $D_z = \delta D_t + \varepsilon D_x$, δ, ε are constants and $a(x, t)$, $b(x, t)$ are arbitrary functions.

Lemma 2.

$$(D_xD_tf' \cdot f')ff - f'f'(D_xD_tf \cdot f) = 2D_x(D_tf' \cdot f) \cdot ff' \tag{7a}$$

$$(D_x^2f' \cdot f')ff - f'f'(D_x^2f \cdot f) = 2D_x(D_xf' \cdot f) \cdot ff' \tag{7b}$$

$$(D_x^4f' \cdot f')ff - f'f'(D_x^4f \cdot f) = 2D_x(D_x^3f' \cdot f) \cdot ff' + 6D_x(D_x^2f' \cdot f) \cdot (D_xf \cdot f') \tag{7c}$$

$$D_t(D_x^2f \cdot f') \cdot ff' = D_x[(D_xD_tf \cdot f') \cdot ff' - (D_tf \cdot f') \cdot (D_xf \cdot f')] \tag{7d}$$

where $f'(x, t)$ and $f(x, t)$ are arbitrary functions.

Making use of lemmas 1 and 2, which can be verified by direct calculation, we have the following theorem.

Theorem. The bilinear equation (2) has the following two-parameter Bäcklund transformation

$$(D_t - aD_x^2 + \xi D_x)f \cdot f' = 0 \tag{8a}$$

$$(-aD_xD_t + D_x^3 + \xi aD_x^2 + D_x - \xi^2 D_x - \eta a)f \cdot f' = 0 \tag{8b}$$

where $a^2 = -3$, ξ, η are two arbitrary parameters and f' and f are two solutions of equation (2).

Proof. Let f and f' be solutions of equation (2). If we can find two equations which relate f with f', and satisfy

$$P \equiv f'f'[(D_t^2 - D_x^2 - D_x^4)f \cdot f] - ff[(D_t^2 - D_x^2 - D_x^4)f' \cdot f'] = 0 \tag{9}$$

this is then a Bäcklund transformation. Following the method of Hirota (1980), we now show that equations (8) are indeed a BT for equation (2).

Making use of equations (7*a*), (7*b*) and (7*c*), P can be rewritten as

$$P = -2D_t(D_xf' \cdot f) \cdot ff' + 2D_x[(D_x + D_x^3)f' \cdot f] \cdot ff' + 6D_x(D_x^2f' \cdot f) \cdot (D_xf \cdot f'). \tag{10}$$

Using equations (6*a*), (8*b*), we have

$$P = 2D_t(D_x f\cdot f')\cdot ff' + 2D_x[(-aD_xD_t + \xi a D_x^2 - \xi^2 D_x + \eta a) f\cdot f']\cdot ff'$$
$$+6D_x(D_x^2 f'\cdot f)\cdot(D_x f\cdot f') \tag{11}$$

and from equations (6*b*), (7*d*) and (8*a*), we obtain

$$P = 2D_t(D_x f\cdot f')\cdot ff' + 2[-aD_t(D_x^2 f\cdot f')\cdot ff' + aD_x(D_t f\cdot f')\cdot(D_x f\cdot f')]$$
$$+2\xi D_x[(D_x - aD_x^2)f\cdot f']\cdot ff' + 6D_x(D_x^2 f'\cdot f)\cdot(D_x f\cdot f')$$
$$= 2D_x[(D_x - aD_x^2 + \xi D_x)f\cdot f']\cdot ff' - 2D_t(\xi D_x f\cdot f')\cdot ff'$$
$$+2D_x[(aD_t + 3D_x^2)f\cdot f']\cdot(D_x f\cdot f') + 2a\xi D_x(D_x f\cdot f')\cdot(D_x f\cdot f')$$
$$+2\xi D_x(D_t f\cdot f')\cdot ff'$$

and, finally,

$$P = 2D_x[(aD_t + 3D_x^2 + a\xi D_x)f\cdot f']\cdot(D_x f\cdot f') \tag{12}$$

which vanishes for $a^2=-3$ by virtue of equation (8*a*). Thus we have proved that equations (8) with $a^2=-3$ constituted a BT for equation (2).

Let $w = \int_{-\infty}^{x} u\,\mathrm{d}x'$; equation (1) can be transformed into

$$w_{tt} - w_{xx} - 6w_x w_{xx} - w_{xxxx} = 0. \tag{13}$$

We introduce the following new variables

$$\phi = \log(f'/f) \qquad \rho = \log(f'f). \tag{14}$$

All the terms of equations (8) can be written by these variables then equations (8) reduce to

$$\phi_t - a\rho_{xx} - a(\phi_x)^2 + \xi\phi_x = 0 \tag{15a}$$

$$-a\rho_{xt} - a\phi_x\phi_t + \phi_{xxx} + 3\phi_x\rho_{xx} + (\phi_x)^3 + a\xi\rho_{xx} + a\xi(\phi_x)^2 + \phi_x - \xi^2\phi_x + \eta a = 0. \tag{15b}$$

Noting that

$$\phi_x = \tfrac{1}{2}(w' - w) \qquad \rho_x = \tfrac{1}{2}(w' + w) \tag{16}$$

equations (15) are equivalent to the following well known form BT for equation (13) (Tu 1981)

$$B_{\xi,\eta}\colon (w'-w)_t - a(w'+w)_{xx} - a(w'-w)(w'-w)_x + \xi(w'-w)_x = 0 \tag{17a}$$

$$-a(w'+w)_t + a\xi(w'-w)^2 + 3(w'-w)(w'+w)_x + (w'+w)_{xx}$$
$$+(w'-w)^3 + a\xi(w'+w)_x + (w'-w) - \xi^2(w'-w) + \eta a = 0 \tag{17b}$$

where w' and w are two solutions of equations (13), ξ, η two arbitrary parameters and $a^2=-3$.

We now use the arbitrary parameters of this BT to derive the soliton solution. First, let $w=0$ (it is the trivial solution of equation (13)), and thus reduce the BT (17) to

$$w'_t - aw'_{xx} - aw'w'_x + \xi w' = 0 \tag{18a}$$

$$-aw'_t + 3w'w'_x + (w')^3 + a\xi(w')^2 + w'_{xx} + a\xi w'_x + w' - \xi^2 w' + \eta a = 0. \tag{18b}$$

Supposing the travelling-wave solution of equation (13) is $w' = f(\alpha x - \beta t + c)$, and substituting it into equation (18a), we obtained

$$-\beta f' - a\alpha^2 f'' - af \cdot \alpha f' + \xi\alpha f' = 0 \tag{19}$$

where $f'(\phi)$ denotes the differentiation with respect to ϕ. In particular, if the parameter ξ takes the value β/α, i.e.

$$\beta = \xi\alpha, \tag{20}$$

equation (19) can be reduced to

$$\alpha f'' = -ff'. \tag{21}$$

Integrating equation (21) with a proper integral constant, we obtained

$$2\alpha f' = (2\alpha)^2 - f^2$$

and integrating once more, we have

$$f = 2\alpha \tanh(\alpha x - \beta t + c). \tag{22}$$

Substituting the above equation (22) into (18b) and using the relation $(\tanh x)' = 1 - \tanh^2 x$, we can derive

$$\xi^2 = 1 + 4\alpha^2 \qquad \eta = -4\alpha^2\xi. \tag{23}$$

Now we obtain the solution of equation (17)

$$w = 2\alpha \tanh[\alpha(x - \xi t + x_0)]$$

and thus the soliton solution of the Boussinesq equation (1)

$$u(x, t) = 2\alpha^2 \operatorname{sech}^2[\alpha(x - \xi t + x_0)] \tag{24}$$

where α and x_0 are arbitrary constants and $\xi^2 = 1 + 4\alpha^2$.

It is interesting to note that this soliton solution is very similar to the following one

$$u(x, t) = 2\alpha^2 \operatorname{sech}^2[\alpha(x - \zeta t + x_0)] \qquad \zeta = 4\alpha^2 \tag{25}$$

which is the soliton solution of the Korteweg–de Vries (KdV) equation $u_t + 6uu_x + u_{xxx} = 0$; the velocity, amplitude and breadth of the two waves are dependent upon the same parameter α.

The nonlinear superposition formula of equation (1) can also be derived from the BT (17). We first change equation (17a) into

$$(w' - w)_t = a[(w' + w)_x + \tfrac{1}{2}(w' - w)^2 - \xi(w' - w)]_x \tag{26}$$

where ξ denotes ξ/a, still an arbitrary parameter, and represent it symbolically by $w_0 \xrightarrow{\xi} w_1$ where the solution w_1 is derived from the solution w_0 under the BT (17) with a parameter ξ. Because of the commutability of Bäcklund transformations, we have

$$w_0 \xrightarrow{\xi_1} w_1 \xrightarrow{\xi_2} w_3 \qquad w_0 \xrightarrow{\xi_2} w_2 \xrightarrow{\xi_1} w_3 \tag{27}$$

and thus

$$(w_1 - w_0)_t = a[(w_1 + w_0)_x + \tfrac{1}{2}(w_1 - w_0)^2 - \xi_1(w_1 - w_0)]_x \tag{28a}$$

$$(w_2 - w_0)_t = a[(w_2 + w_0)_x + \tfrac{1}{2}(w_2 - w_0)^2 - \xi_2(w_2 - w_0)]_x \tag{28b}$$

$$(w_3 - w_1)_t = a[(w_3 + w_1)_x + \tfrac{1}{2}(w_3 - w_1)^2 - \xi_2(w_3 - w_1)]_x \tag{28c}$$

$$(w_3 - w_2)_t = a[(w_3 + w_2)_x + \tfrac{1}{2}(w_3 - w_2)^2 - \xi_1(w_3 - w_2)]_x. \tag{28d}$$

From equations (28), we have

$$[2(w_1-w_2)_x+(w_1^2-w_2^2)+(w_2-w_1)(w_0+w_3)-\xi_1(w_1+w_2)$$
$$+\xi_2(w_1+w_2)+(w_0+w_3)(\xi_1-\xi_2)]_x=0 \qquad (29)$$

and thus, by integration of equation (29) with λ being the integral constant, we obtained the following nonlinear superposition formula:

$$w_3=-w_0+\frac{1}{w_2-w_1+\xi_1-\xi_2}[\lambda+2(w_2-w_1)_x+(w_2^2-w_1^2)+(w_2+w_1)(\xi_1-\xi_2)]. \qquad (30)$$

Finally, using the arbitrary parameter we derive, from the BT (17), the infinitely many conservation laws of the form

$$\frac{\partial D_n}{\partial t}+\frac{\partial F_n}{\partial x}=0 \qquad n=1,2,3,\ldots \qquad (31)$$

where D_n, F_n are relative to the solutions of equation (1). Let $v=w'-w$, $\eta=0$; we rewrite the BT (17) in the following form

$$w'_t=aw'_{xx}+\tfrac{2}{3}av_{xx}-aw'_x+\xi v_x+avw'_x+\tfrac{1}{6}av^3-\tfrac{1}{2}\xi v^2-\xi w'_x+\tfrac{1}{6}av+\tfrac{1}{6}\xi^2av=0 \qquad (32a)$$

$$v_t=[\tfrac{1}{2}av^2+2aw'_x-av_x-\xi v]_x. \qquad (32b)$$

Substituting $v=\sum_{n=1}^{\infty}f_n\xi^{-n}$ into equation (32b) and equating the coefficients for the higher powers of ξ^{-1}, we find the infinite number of the conservation laws:

$$\begin{aligned}(f_1)_t&=(-af_{1x}-f_2)_x\\ (f_n)_t&=\Big(-af_{nx}+\frac{a}{2}\sum_{k+l=n}f_kf_l-f_{n+1}\Big)_x \qquad n\geqslant 2.\end{aligned} \qquad (33)$$

Substituting equation $v=\sum_{n=1}^{\infty}f_n\xi^{-n}$ into equation (32a) too, we can derive the explicit expressions of f_n one by one

$$f_1=2aw'_x \qquad (34)$$

$$f_2=6w'_{xx}-2aw'_t \qquad (35)$$

$$f_3=2aw'_x-4aw'_{xxx}-12w'_{tx} \qquad (36)$$

$$f_4=36w'_xw'_{xx}+18w'_{xx}+12aw'_xw'_t+16aw'_{txx}-2aw'_t \qquad (37)$$

and in general,

$$f_{n+2}=-2a(f_{n+1})_x+4f_{nxx}+(6w'_x+1)f_n$$
$$+a\sum_{k+l=n+1}f_kf_l-6\sum_{k+l=n}f_kf_{lx}+\sum_{k+l+p=n}f_kf_lf_p \qquad n\geqslant 3 \qquad (38)$$

where w' is the solution of equation (13).

The recurrence relation (38) was first obtained by Zakharov (1973) who also proved that the conserved densities $f_1, f_2, \ldots$ are in involution.

The author would like to thank Professor Tu Gui-Zhang for both his guidance and his kindness for allowing him to use some results in his paper (1981). He is also grateful to Associate Professor Chen An-Sheng for frequent, helpful discussion and advice.

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